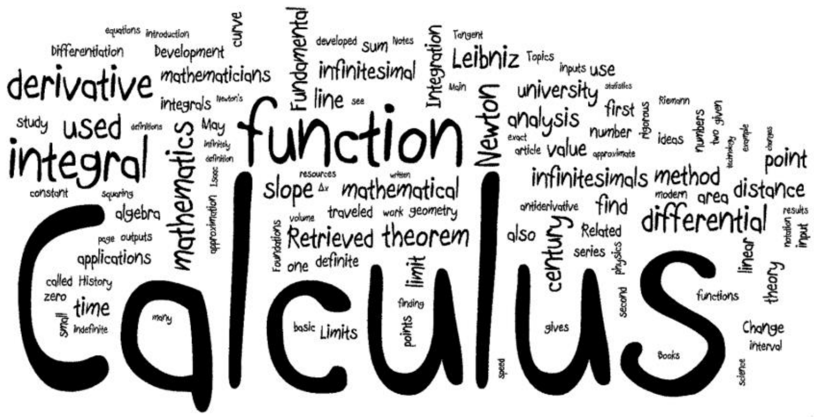
Name: ____

AP Calculus Summer Homework

with Mr. Siepiela & Mr. French



Calculus is both new and old. It's a new way of looking at concepts you have been exploring for years—slope, area, rates, etc. This summer homework is composed of "Problem Sets"—some easy, some hard—to get you thinking about old math concepts in the new ways we are going to talk about them next year. A few tips to get you started:

1) Read each problem well. In addition to questions, many problems contain definitions and explanations. Words that have been *italicized* are often new. Sometimes you will have to use extra resources to find out what these mean.
2) Problem-solving requires persistence as well as ingenuity. If you get stuck, back up and start over. If you have tried multiple ideas and nothing works, bring a record of your work to class so we can discuss ideas. Just because you don't finish a problem does NOT mean your ideas aren't valuable to the discussion.
3) Many of these problems, except where indicated, can be made easier using a graphing calculator. Look at problems in multiple ways—you come up with new ideas looking at a graph rather than looking at an equation.

You are expected to complete ***every problem*** in Problem Sets 1-4 to the best of your ability. Bring all your work, not just the successful attempts, to class in order to contribute to the discussion. There is a challenge Problem Set 5 at the end for those of you that want to push yourselves. It is not required, but we are curious how far some of you can get.

Time Requirement – In order to be completed thoroughly, each Problem Set should take between 1-2 hours to finish. That means the entire assignment should take you **4 – 8 hours** to complete, excluding PS5 which would take extra time. Make sure to plan your time on a calendar so you don't end up waiting until the last minute at the end of the summer!

If you just can't figure something out, use resources such as "the Google" to see if you can find some ideas. There is also a Google Classroom site (code: i6iarp2) you can post questions to. If you exhaust all other options, send an e-mail Mr. Siepiela (wsiepiela@goldercollegeprep.org) or Mr. French (chfrench@goldercollegeprep.org)! We will be frequently available over the summer if you need to come in to talk some problems over or get ideas of what to try next.

These problems are meant to be puzzles. Not all of them have a perfect solution. Have fun with them and try not to get discouraged if you can't figure everything out right away! Learning takes time, especially with calculus. Mr. French and Mr. Siepiela look forward to seeing you in August!

Problem Set 1: Equations & Functions

1. A linear function can be modeled by the formula $y = mx + b$ where m is the slope and b is the y-intercept. If these values are known, the line can be drawn. What is the slope of a quadratic function? Is there a place in $y = ax^2 + bx + c$ where this value can be found?

2. Write $x^2 - 4$ as a product of linear factors. If you graph a linear factor (set equal to y), you get a straight line. Why, when you multiply two linear factors together, is the resulting graph *non-linear*?

3. Find a function f for which $f(x+3)$ is *not* equivalent to $f(x) + f(3)$. Then find an f for which $f(x+3)$ *is* equivalent to $f(x) + f(3)$.

4. Given $f(x) = \sqrt{x}$ and $g(x) = x^2 + 9$, find:
   (a) $f(x-1)$
   (b) $g(2x)$
   (c) $f\big(g(5)\big)$
   (d) $g\big(f(5)\big)$
   (e) $f\big(g(x)\big)$

5. It is evident that $s(x) = \sin(2x)$ is expressed in the form $p\big(q(x)\big)$. In this situation, we say s is a *composite* of p and q. Notice that s is *periodic*. Is $q\big(p(x)\big)$?

6. Find functions p and q that show that $p\big(q(x)\big)$ can be equivalent to $q\big(p(x)\big)$.

7. Given a function f, each solution to the equation $f(x) = 0$ is called a *zero* of f. *Without* using a calculator, find the zeros of the following and be ready to explain your work:
   (a) $s(x) = \sin 3x$
   (b) $L(x) = \log_5(x-3)$
   (c) $r(x) = \sqrt{2x+5}$
   (d) $p(x) = x^3 - 4x$

8. A zero of a function is sometimes called a *root* of that function.
   (a) Find the roots of $f(x) = x^4 + 3x^2 - 4$. Use the calculator to assist you.
   (b) The function f defined by $f(x) = x^2 - 2x + 1$ has a double root. Explain.

9. Consider the cubic graph $y = 3x^2 - x^3$.
   (a) Write $3x^2 - x^3$ in its factored form.
   (b) Use this factored form to explain why the graph lies below the x-axis only when $x > 3$, and why the origin is therefore an *extreme point* on the graph.

Problem Set 2: Limits

1. *Sequences* are lists of numbers in a special order determined by a relationship or equation. For example, $x_n = \frac{1}{n}$. Each number in a sequence is called a *term* and is designated by a *subscript*. For example, the 4th term in the sequence x_n is x_4, where $n = 4$. Thus, in the above sequence, $x_4 = \frac{1}{4}$. Calculate $x_5, x_6, x_{15}, x_{140}$ and x_{14523}. What do you notice about your answers as the value of n gets larger? Is there a value of n that makes the sequence reach a "final" number? Explain.

2. When the terms in a sequence approach a certain "limiting" value, we call this the *limit* of the sequence. In the above example, as n got larger and larger (approaching infinity), the terms got smaller and smaller (approaching a certain value L). Using mathematical notation, we write this situation as $\lim_{n\to\infty} \frac{1}{n} = L$. This is pronounced as "the limit of one over n as n approaches infinity equals L". Find the limit of the sequence of $x_n = n \sin \frac{1}{n}$ as n gets larger and larger, or "approaches infinity".

3. For $p = 0.0001$, it is true that $\left(\frac{5}{6}\right)^n < p$ for all sufficiently large values of n. How large is "sufficiently large"? If p were even smaller, is there still a sufficiently large value of n that would satisfy the inequality? If $p = 0$, is there a sufficiently large value of n such that $\left(\frac{5}{6}\right)^n = p$? Explain.

4. Some sequences can be *recursive*, meaning a certain term, x_n, depends upon the previous term, x_{n-1}. Consider the sequence defined recursively by $x_n = \sqrt{\sqrt{1996 x_{n-1}}}$ and $x_0 = 1$. Starting with x_1, calculate the first 10 terms of this sequence, and decide whether it has a limit.

5. The idea of limits applies to more than just sequences. Consider the function $f(x) = \frac{x^2+x-2}{x-1}$. What is the difficulty in evaluating $f(1)$? We can use the idea of limits to assist us. Calculate $f(0.5)$, $f(0.9)$, $f(0.99)$, and $f(0.999)$. What is $\lim_{x\to 1} f(x)$? Is your answer the same approaching $x = 1$ from values larger than 1?

6. The function $g(x) = \left(1 + \frac{1}{x}\right)^x$ is an approximation of a formula used in finance to calculate interest where x represents the frequency at which interest is compounded, or added to the original amount. Find $\lim_{x\to\infty} g(x)$ by finding $g(1), g(12), g(365)$, and $g(31536000)$. This limit is actually so important that a special letter is reserved for it (like how π is reserved for 3.1415...). What is this letter?

Problem Set 3: Rates

1. An old bathtub has two faucets, one for hot and one for cold. Running at full, the cold faucet can fill the tub in 30 minutes. Running at full, the hot faucet can fill the tub in an hour. If both faucets are turned on full at the same time, how long would it take for the tub to fill? Why is the answer not as simple as averaging the two times (spoiler alert: the answer is *not* 45 minutes)?

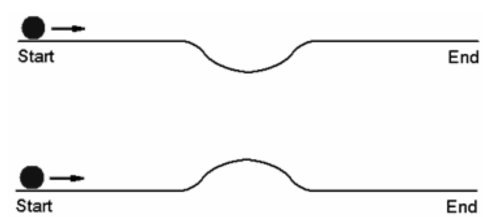
2. A person travels from city A to city B with a speed of 40 mph and returns with a speed of 60 mph. What is his average round-trip speed?

3. The Eiffel Tower has a mass of 10,000,000 kg. A 100:1 scale model of the tower made from the same material will have a mass of what?

4. Mr. Siepiela is returning home at a speed of 2 mph with his dog Karl Barx. He unleashes Barx when they are still 3 miles from his house. Barx happily begins running back and forth between the house and his master with a constant speed of 3 mph. Barx does not waste any time while turning around. By the time Mr. Siepiela reaches home, how many miles has Barx run?

5. Two trains are moving toward each other with speeds of 17 mph and 43 mph. How far apart are they 1 minute before they pass each other?

6. Two marbles roll along two horizontal tracks from the same starting point with equal initial speed. One track has a dip, and the other has a bump of the same shape as shown in the figure to the right. Which marble wins?



7. The point $(1, 1)$ is on the graph of $y = x^3$. Find coordinates for another point *on the graph* and very close to $(1, 1)$. Find the slope of the line that goes through these points. Explain how this slope is related to the value of $\lim_{x\to 1} \frac{x^3-1}{x-1}$.

8. An empty, spherical fish bowl is being filled with water. If water is being added to the bowl at a constant rate, describe the rate of change of the height of the water at various levels within the sphere and explain your reasoning.

9. Suppose a scientist is collecting data on bacteria growth in a new environment. She records the number of bacteria every day for 7 days. She does some calculations with the data and finds that the rate of change each day is positive. However, because of the new environment, the *rate of change of the rate of change* is negative. What would this graph look like? Sketch a sample graph assuming the initial amount of bacteria was y_i and the final amount after 7 days was y_f.

Problem Set 4: Modeling

1. Draw a graph that displays plausibly how the temperature changes during a 48-hour period at a desert site. Assume that the air is still, the sky is cloudless, the sun rises at 7am and sets at 7pm. Be prepared to explain all of the details of your graph. Design a function that matches your graph as closely as possible. If you use a piecewise function, be sure to specify the domain of each piece.

2. Garbanzo bean cans usually hold 4000 cubic centimeters (4 liters). It seems likely that the manufacturers of these cans have chosen the dimensions so that the material required to enclose 4000 cc is as small as possible. Let's find out what those optimal dimensions are.
   (a) Find an example of a right circular cylinder whose volume is 4000 cc. Calculate the total surface area of your cylinder, in square cm.
   (b) Express the height and surface area of such a cylinder as a function of its radius r.
   (c) Find the value of r that gives a cylinder of volume 4000 cc the smallest total surface area that it can have, and calculate the resulting height.

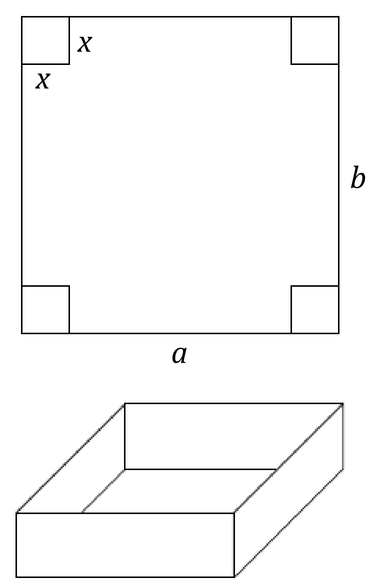
3. A ball falls from a tall building and its altitude is given by $y = 784 - 16t^2$ where y is in feet and t is in seconds. What is the altitude of the ball when $t = 2$s? What is the altitude of the ball a little later at time $t = 2 + k$? How much altitude has the ball lost during this tiny k-second interval? At what *rate* is the ball losing altitude during this interval in terms of k?

4. Based on the previous problem, evaluate $\lim_{k\to 0} \frac{y(2+k)-y(2)}{k}$, recalling that $y(t) = 784 - 16t^2$ is the altitude of the ball at time t. What is the meaning of this limiting value in this story?

5. Copper wire is being cooled in a physics experiment, which is designed to study the extent to which lowering the temperature of a metal improves its ability to conduct electricity. At a certain stage of the experiment, the wire has reached a temperature at which its conductance would increase 28.8 *mhos* for each additional degree dropped. At this moment, the temperature of the wire is decreasing at 0.45 degrees per second. At what rate—in *mhos* per second—is the conductance of the wire increasing at this moment?

6. A hiker started to climb up the hill at 6:00 a.m. and either kept climbing up or rested at some place(s). He reached the top at 6:00 p.m. He rested there for the next 12 hours. Next day at 6:00 a.m., he began to travel down the same path. He either moved downward or rested at some place(s). For the up and down trips, how many times was he at the same place at the same time?

7. A box with an open top is to be constructed from a rectangular piece of cardboard with dimensions b = 12 in. by a = 20 in. by cutting out equal squares of side x at each corner and then folding up the sides as in the figure (see figure to the right). Express the volume V of the box as a function of x. What are the domain and range of this function? What do they represent?

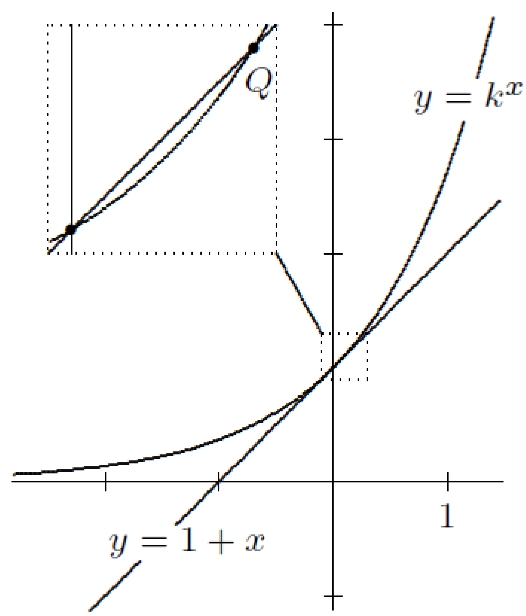


Problem Set 5: Beast Mode

1. Below is a chart of the total amount of United States' GDP, or *gross domestic product* from 2003 to 2008. GDP is a measurement, in dollars, of how much the US economy is producing at that time and is used by economists to determine the "health" of our economy. The more we produce, the better the economy is. In the chart, the GDP is clearly increasing from 2003 to 2008, so some claim this is proof that our economy was healthy until "Obama ruined it" when he took office. Using the data provided, explain why the economy was not as healthy as people claimed and how the data foreshadows the 2008 financial collapse and recession. (Hint: Economists calculate the change in GDP from year to year as *percent change*, so you should too. Why is percent change a more accurate way of describing growth in our economy rather than simple subtraction?)

Year	2003	2004	2005	2006	2007	2008
GDP (in trillions of $)	$11.5	$12.3	$13.1	$13.9	$14.5	$14.7

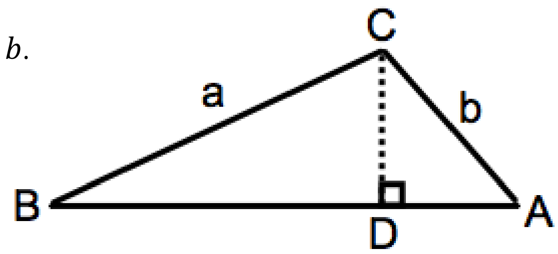
2. The *slope of the curve* $y = 2^x$ at its *y*-intercept is slightly less than 0.7, while the slope of the curve $y = 3^x$ at its *y*-intercept is nearly 1.1. This suggests that there is a number *b* for which the slope of the curve $y = b^x$ is <u>exactly</u> 1 at its *y*-intercept. The figure shows the line $y = 1 + x$, along with the graph of $y = k^x$, where *k* is slightly smaller than the special number *b*. The curve crosses the line at (0, 1) and (as the magnified view shows) at another point *Q* nearby in the first quadrant. Given the *x*-coordinate of *Q*, it is possible to calculate *k* by just solving the equation $k^x = 1 + x$ for *k*. Do so when $x = 0.1$, when $x = 0.01$, and when $x = 1/n$. The last answer expresses *k* in terms of *n*; evaluate the limit of this expression as *n* approaches infinity, and deduce the value of *b*. What happens to *Q* as *n* approaches infinity?



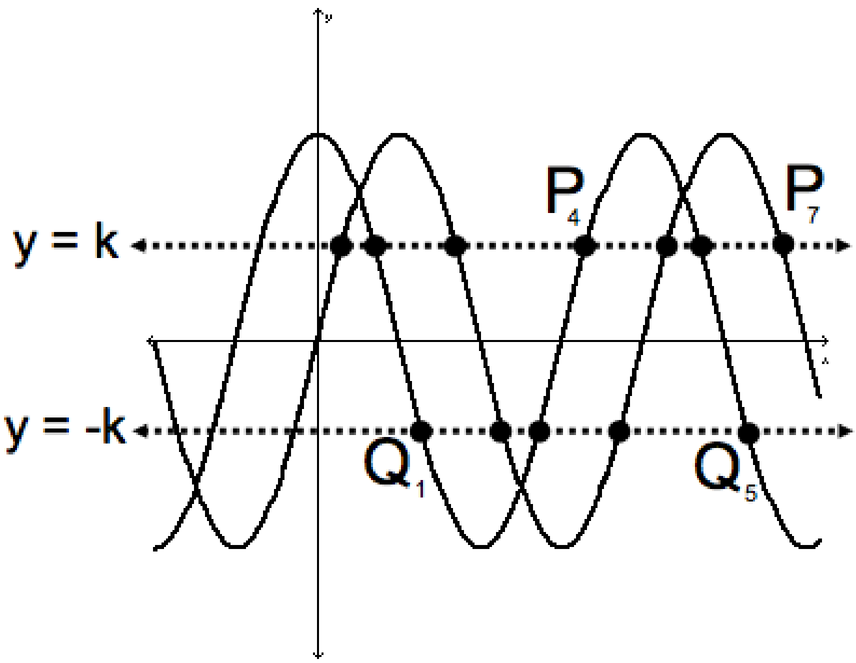
3. *The Babylonian algorithm*. Calculate a few terms of the sequence defined by the *seed value* $x_0 = 1$ and the recursion $x_n = \frac{1}{2}\left(x_{n-1} + \frac{5}{x_{n-1}}\right)$. Find $\lim_{n\to\infty} x_n$, and thereby discover what this sequence was designed to do (*circa* 1600 BCE).

4. It is a fact that the square root of 2 is the same as the fourth root of 4—in other words, $2^{\frac{1}{2}} = 4^{\frac{1}{4}}$. Thus, the graph of $y = x^{\frac{1}{x}}$ goes through two points that have the same y-coordinate at $x = 2$ and $x = 4$. Between these points is a maximum y-value for the graph. Find the x-value associated with this maximum to four decimal places.

5. In the diagram to the right, CD is the altitude from C.
   (a) Simplify the expression $(a \sin B)^2 + (a \cos B)^2$ and discuss its relevance to the diagram.
   (b) Express AD in terms of angle B, side a, and side b.
   (c) Express angle C in terms of angle B, side a, and side b.
   (d) For what value(s) of angle B will $a = b$? Explain.



6. The graph below shows the two curves $sin(x)$ and $cos(x)$ intersecting the lines $y = k$ and $y = -k$ for some $k \ni (0, 0.6)$. The points P_m are the points in the first quadrant where the line $y = k$ intersects a curve. The points Q_n are the points in the fourth quadrant where the line $y = -k$ intersects a curve. Some of these points have been labeled for you to clarify. P_1 has the coordinates (θ, k).



   (a) Express the coordinates of the following points in terms of θ and k: P_2, P_3, P_5, P_6.
   (b) Express the coordinates of the following points in terms of θ and k: Q_1, Q_2, Q_4, Q_5.
   (c) Create a function, piecewise or other, in terms of m that yields the correct coordinates of P_m.
   (d) Create a function, piecewise or other, in terms of n that yields the correct coordinates of Q_n.
   (e) The value of k was restricted to $(0, 0.6)$. State the obvious reason why numbers greater than 1 were excluded.
   (f) Only one value of k caused values from 0.6 to 1 to be excluded. What is that value? Why would it be excluded?